

Topological aspects of irredundant intersections of ideals and valuation rings

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Abstract An intersection of sets $A = \bigcap_{i \in I} B_i$ is irredundant if no B_i can be omitted from this intersection. We develop a topological approach to irredundance by introducing a notion of a spectral representation, a spectral space whose members are sets that intersect to a given set A and whose topology encodes set membership. We define a notion of a minimal representation and show that for such representations, irredundance is a topological property. We apply this approach to intersections of valuation rings and ideals. In the former case we focus on Krull-like domains and Prüfer v -multiplication domains, and in the latter on irreducible ideals in arithmetical rings. Some of our main applications are to those rings or ideals that can be represented with a Noetherian subspace of a spectral representation.

Keywords • Zariski-Riemann space • valuation ring • Krull domain • Prüfer domain • Prüfer v -multiplication ring • spectral space

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1 Introduction

The goal of this article is to develop a topological framework for recognizing and dealing with an irredundant infinite intersection of ideals, subrings, submodules, even sets. While our main interest here is in the intersection of valuation rings, we include one application to the intersection of irreducible ideals in arithmetical rings to illustrate how the framework applies in a different setting. A key requirement for our point of view is that the objects from which the intersection is formed be drawn from a spectral space whose topology encodes set membership. The Zariski

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topology on the set of irreducible ideals of an arithmetical ring provides one such context, while the inverse topology on the Zariski-Riemann space of valuation rings of a field is another. Several other contexts to which our approach applies, and which we do not pursue, are given in Example 2.2.

Irredundance of intersections of valuation rings is often a consequential and special phenomenon. For example, if F/k is a finitely generated field extension of transcendence degree one with k algebraically closed in F , and \mathfrak{X} is the set of all valuation rings containing k and having quotient field F , then $k = \bigcap_{V \in \mathfrak{X}} V$ and this intersection is irredundant. Thus \mathfrak{X} is the unique representation of k as an (irredundant) intersection of valuation rings in \mathfrak{X} . This is a consequence of Riemann's Theorem for projective curves and is closely related to the Strong Approximation Theorem for such curves [25, Theorem 2.2.13]. If, however, F/k has transcendence degree > 1 , then k can still be represented by an irredundant intersection of valuation rings (albeit by very specially selected subsets of \mathfrak{X}), but there exist infinitely many such representations. Such examples can be constructed along the lines of [41, Example 6.2].

In general, the existence, much less uniqueness, of an irredundant representation of a ring can only be expected under circumstances where “few” valuation rings are needed to represent the ring. For example, Krull domains can all be represented by an irredundant intersection of valuation rings, but this ultimately depends on the fact that they can be represented by a finite character intersection of valuation rings; see Section 5. On the other hand, if F/k is a function field in more than one variable and k is existentially, but not algebraically, closed in F , and A is the intersection of all the valuation rings in F/k having residue field k , then no representation X of A as an intersection of valuation rings contains an irredundant member; i.e., any member of X can be omitted and the intersection will remain A ; see [43, Theorem 4.7]. This last example is even a Prüfer domain and hence has the property that every valuation ring between A and its quotient field is a localization of A . Thus even for classes of rings whose valuation theory is explicitly given by their prime spectra, intersections of valuation rings can behave in complicated ways.

In Section 3 we develop a topological approach to these issues for intersections of sets, where the sets themselves can be viewed as points in a spectral space. The prime ideals of a ring or the valuation rings of a field comprise such sets when viewed with the appropriate topologies, but also so do the irreducible ideals in an arithmetical ring. Throughout this article we are particularly interested in Noetherian spectral spaces, and in Section 2 we work out some of the properties of these spaces when viewed under the inverse or patch topologies. (These topologies are reviewed in Section 2.) Krull domains, and generalizations of these rings of classical interest, can be represented by intersections of valuation rings drawn from a Noetherian subspace of a spectral space, and we apply the results from Sections 2 and 3 in Sections 5 and 6 to intersections of valuation rings from a Noetherian subspace of the Zariski-Riemann space of a field.

Sections 4-7 contain the main applications of the article. Section 4 applies the abstract setting of spectral representations to the Zariski-Riemann space of a field. This section recasts the abstract approach in Section 3 into a topological framework

for working with irredundance in intersections of valuation rings. Section 5 specializes the discussion to the Krull-like rings of classical interest and recaptures the representation theorems for these rings. A feature throughout Sections 4 and 5 that is afforded by the abstract approach of spectral representations is that intersections of valuation rings can be considered relative to a subset of the ambient field. The motivation for this comes from the articles [1, 30, 38, 42, 46]. In these studies, one considers integrally closed domains A between a given domain and overring, e.g., between $\mathbb{Z}[T]$ and $\mathbb{Q}[T]$. In such cases A is an intersection of $\mathbb{Q}[T]$ and valuation rings not containing $\mathbb{Q}[T]$. Since $\mathbb{Q}[T]$ can be viewed as always present in these representations, it is helpful then to consider representations of a ring A of the form $A = (\bigcap_{V \in X} V) \cap C$, where C is a fixed ring. The approach provided by Section 3 makes it easy to incorporate a fixed member C of the representation into such a picture, regardless of whether C is a ring or simply a set.

The already well-understood theory of irredundance for Prüfer domains also can be recovered from our framework, and this is done in Section 6 in the more general setting of Prüfer v -multiplication domains. We consider existence and uniqueness for irredundant representations of such domains, with special emphasis on the case in which the space of t -maximal ideals is Noetherian. When restricted to a Prüfer domain A , these results specialize to a topological characterization of the property that every overring of A is an irredundant intersection of the valuation rings that are minimal over it.

In order to help justify the generality of the approach Section 3, we show in Section 7 how the topological framework can be applied to the study of irredundant intersections of irreducible ideals in arithmetical rings. We show in particular how intersection decomposition results involving such ideals can be recovered from our point of view. This section is independent of the valuation-theoretic Sections 4–6.

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2 Spectral spaces

A *spectral space* is a T_0 topological space having (a) a basis of quasicompact open sets closed under finite intersections, and (b) the property that every irreducible closed subset has a unique generic point, i.e., a point whose closure is the irreducible closed set. By a theorem of Hochster [33, Corollary, p. 45], a topological space X is spectral if and only if X is homeomorphic to the prime spectrum of a ring. In the setting of this paper, it is mostly the topological features of spectral spaces that are needed rather than the connection with prime spectra of rings.

A spectral space X admits two other well-studied topologies that are useful in our context. The *inverse topology* on X has as a basis of closed sets the subsets of X that are quasicompact and open in the spectral topology. By an *inverse closed* subset of X we mean a subset that is closed in the inverse topology. The *patch topology* has as a basis of open sets the sets of the form $U \cup V$, where U is open and quasicompact

in the spectral topology and V is the complement of a quasicompact open set. These basic open sets are also closed, so that the patch topology is zero-dimensional and Hausdorff. A *patch* in X is a set that is closed in the patch topology. In this section we denote the closure of a subset Y of X in the spectral topology as \overline{Y} , and the closure of Y in the patch topology as \widetilde{Y} .

The patch topology refines both the spectral and inverse topologies. This can be made more precise using the *specialization order* of the spectral topology: If $x, y \in X$, then $x \leq y$ if and only if $y \in \overline{\{x\}}$ in the spectral topology. With this order in mind, we define for $Y \subseteq X$,

$$\begin{aligned} \uparrow Y &= \{x \in X : x \geq y \text{ some } y \in Y\} \quad \text{and} \quad \downarrow Y = \{x \in X : x \leq y \text{ some } y \in Y\}, \\ \text{Min } Y &:= \{y \in Y : y \text{ is minimal in } Y \text{ with respect to } \leq\}, \\ \text{Max } Y &:= \{y \in Y : y \text{ is maximal in } Y \text{ with respect to } \leq\}. \end{aligned}$$

Proposition 2.1. *Let X be a spectral space with specialization order \leq . Then*

- (1) *X with the inverse topology is a spectral space whose specialization order is the reverse of that of (X, \leq) .*
- (2) *X with the patch topology is a spectral space, and in particular a compact Hausdorff zero-dimensional space.*
- (3) *For each $Y \subseteq X$, $\overline{Y} = \uparrow(\widetilde{Y})$ and the closure of Y in the inverse topology is $\downarrow(\widetilde{Y})$.*
- (4) *If Y is a patch in X , then the following statements hold.*
 - (a) *Y is spectral in the subspace topology.*
 - (b) *For each $y \in Y$ there exists $m \in \text{Min } Y$ with $m \leq y$.*
 - (c) *The patch and spectral topologies agree on $\text{Min } Y$.*

Proof. Statement (1) can be found in [33, Proposition 8]; statement (2) can be deduced from [33, Section 2]. Statement (3) is a consequence of [33, Corollary, p. 45] and (1). Statement (4)(a) follows from [33, Proposition 9]. Statement (4)(b) now follows from (a), since a spectral space has minimal elements. Finally, the spectral and patch topologies agree on the set of minimal elements of a spectral space [49, Corollary 2.6], so (4)(c) follows from (4)(a). \square

We give now a list of examples of some spectral spaces in our context. We only use a few of these examples in what follows, but the intersection representation theory developed in the next section applies to all of them. As we indicate, several of these examples have appeared in the literature before, but with different proofs than what we give here. Our approach is inspired by a theorem of Hochster [33, Proposition 9] that a topological space is spectral if and only if it is homeomorphic to a patch closed subset of a power set endowed with the hull-kernel topology. Interestingly, inspection of Zariski and Samuel's proof in [51] that the Zariski-Riemann space \mathfrak{X} of a field is quasicompact shows that although their work predated the notion of spectral spaces, what is proved there is that \mathfrak{X} is a patch closed subset of a certain spectral space, and hence from their argument can be deduced the fact that \mathfrak{X} is spectral.

To formalize the setting of the example, let S be a set. We denote by 2^S the power set of S endowed with the hull-kernel topology having as an open basis the sets

of the form $\mathcal{U}(F) := \{B \subseteq S : F \not\subseteq B\}$, where F is a finite subset of S . The complement of $\mathcal{U}(F)$ is denoted $\mathcal{V}(F)$; i.e., $\mathcal{V}(F) = \{V \subseteq S : F \subseteq V\}$. Then the sets $\mathcal{U}(F)$ are quasicompact and 2^S is a spectral space; cf. [33, Theorem 8 and Proposition 9]. Thus by Proposition 2.1, to show that a collection X of subsets of S is a spectral space in the subspace topology, it is enough to show that X is patch closed in S . Specifically, what must be shown is that X is an intersection of sets of the form $\mathcal{V}(F_1) \cup \dots \cup \mathcal{V}(F_n) \cup \mathcal{U}(G)$, where F_1, \dots, F_n, G are finite subsets of S . This is done in each case by encoding the question of whether a given subset of S satisfies a first-order property in the relevant language into an assertion about membership in a set of the form $\mathcal{V}(F_1) \cup \dots \cup \mathcal{V}(F_n) \cup \mathcal{U}(G)$. This amounts in most cases to rewriting a statement of the form “ $p \rightarrow q$ ” as “(not p) or q .” Because the goal is to produce patch closed subsets, statements involving universal quantifiers (which translate into intersections) are more amenable to this approach than statements involving existential quantifiers (which translate into infinite unions).

To clarify terminology, when R is a ring, the *Zariski topology* on a collection X of ideals of R is the hull-kernel topology defined above; i.e., it is simply the subspace topology on X inherited from 2^R . This agrees with the usual notion of the Zariski topology on $\text{Spec } R$. However, when S is a ring and X is a collection of subrings of S , then the *Zariski topology* on X is the inverse of the hull-kernel topology; i.e., it has an open basis consisting of sets of the form $\mathcal{V}(G)$, where G is a finite subset of S . Despite the discrepancy, it is natural to maintain it in light of the fact that when R is a subring of a field F , then with these definitions, $\text{Spec } R$ with the Zariski topology is homeomorphic to the space $\{R_P : P \in \text{Spec } R\}$ of subrings of F with the Zariski topology. This discrepancy, which is due to Zariski, also allows for an identification between projective models and projective schemes; cf. [51] for the notion of a projective model.

Example 2.2. (1) *The set of all proper ideals of a ring R is a spectral space in the Zariski topology.* The set of proper ideals in R is precisely the patch closed subset of 2^R given by

$$X_1 = \mathcal{U}(1) \cap \left(\bigcap_{a,b \in R} \mathcal{U}(a,b) \cup \mathcal{V}(a+b) \right) \cap \left(\bigcap_{a,r \in R} \mathcal{U}(a) \cup \mathcal{V}(ra) \right).$$

(2) *If R is a ring, the set of all submodules of an R -module is a spectral space in the Zariski topology.* An easy modification of (1) shows this to be the case.

(3) *The set of all radical ideals of a ring R is a spectral space in the Zariski topology.* The set of radical ideals is precisely the patch closed subset of 2^R given by

$$X_3 = X_1 \cap \left(\bigcap_{a \in R, n > 0} (\mathcal{U}(a^n) \cup \mathcal{V}(a)) \right).$$

(4) *If R is a ring such that $aR \cap bR$ is a finitely generated ideal of R for all $a, b \in R$, then the set of all proper strongly irreducible ideals is a spectral space in the Zariski topology.* Recall that an ideal I of R is strongly irreducible if whenever $J \cap K \subseteq I$, then $J \subseteq I$ or $K \subseteq I$; equivalently, I is strongly irreducible if and only if whenever

$a, b \in R$ and $aR \cap bR \subseteq I$, it must be that $a \in I$ or $b \in I$. Thus the set of strongly irreducible proper ideals in R is given by

$$X_4 = X_1 \cap \left(\bigcap_{a,b \in R} (\mathcal{U}(aR \cap bR) \cup \mathcal{V}(aR) \cup \mathcal{V}(bR)) \right).$$

By assumption, for each $a, b \in R$, $aR \cap bR$ is a finitely generated ideal of R , so the set $\mathcal{U}(aR \cap bR)$ is quasicompact and open. Therefore, X_4 is a patch closed subset of 2^R . This example will be used in Section 7.

(5) (Finocchiaro [8, Proposition 3.5]) *Let $R \subseteq S$ be an extension of rings. The set of rings between R and S with the Zariski topology is a spectral space.* The set of rings between R and S is given by the patch closed set

$$X_5 = \left(\bigcap_{r \in R} \mathcal{V}(r) \right) \cap \left(\bigcap_{a,b \in S} \mathcal{U}(a,b) \cup \mathcal{V}(a+b,ab) \right).$$

The Zariski topology on X_5 is the inverse topology of the subspace topology on X_5 inherited from 2^S , so by Proposition 2.1, X_5 is spectral in the Zariski topology.

(6) (Finocchiaro [8, Proposition 3.6]) *Let $R \subseteq S$ be an extension of rings. The set of all integrally closed rings between R and S with the Zariski topology is a spectral space.* Let \mathcal{M} denote the set of monic polynomials in $S[T]$, and for each $f \in \mathcal{M}$, let $c(f)$ denote the set of coefficients of f . The set of integrally closed rings between R and S is given by the patch closed set

$$X_6 = X_5 \cap \left(\bigcap_{s \in S} \left(\bigcap_{f \in \mathcal{M}, f(s)=0} \mathcal{U}(c(f)) \cup \mathcal{V}(s) \right) \right).$$

As in (5), this implies that X_7 is spectral in the Zariski topology.

(7) (Finocchiaro-Fontana-Spirito [12, Corollary 2.14]) *Let R be a subring of a field F . The set of all local rings between R and F with the Zariski topology is a spectral space.* A ring A between R and F is local if whenever a, b are nonzero elements of R with $1/(a+b) \in R$, we have $1/a \in R$ or $1/b \in R$. Thus the set of all local rings between R and F is given by the patch closed subset

$$X_7 = X_5 \cap \left(\bigcap_{0 \neq a,b \in F} \mathcal{U}(a,b,1/(a+b)) \cup \mathcal{V}(1/a) \cup \mathcal{V}(1/b) \right).$$

As in (5), this implies that X_7 is spectral in the Zariski topology.

(8) *Let A be a subring of a field F . The set of all valuation rings containing A and having quotient field F is a spectral space in the Zariski topology.* This has been proved by a number of authors; see [9, 45] for discussion and references regarding this result. A subring V between A and F is a valuation ring with quotient field F if and only if for all $0 \neq q \in F$, $q \in V$ or $q^{-1} \in V$. Thus with $R = A$ and $S = F$, we use the set X_5 from (5) to obtain the set of valuation rings of F containing A as the patch closed subset of 2^F given by

$$X_8 = X_5 \cap \bigcap_{0 \neq q \in F} \mathcal{V}(q) \cup \mathcal{V}(q^{-1}).$$

As in (5), this implies that X_8 is spectral in the Zariski topology.

For the remainder of the section we focus on Noetherian spectral spaces, since these play a central role in later sections. A topological space is *Noetherian* if its open sets satisfy the ascending chain condition. Rush and Wallace [48, Proposition 1.1 and Corollary 1.3] have shown that a collection Y of prime ideals of a ring R is a Noetherian subspace of $\text{Spec } R$ if and only if for each prime ideal P of R , there is a finitely generated ideal $I \subseteq P$ such that every prime ideal in Y containing I contains also P . Since every spectral space can be realized as $\text{Spec } R$ for some ring R , we may restate this topologically in the following form.

Lemma 2.3. (Rush and Wallace) *Let X be a spectral space, and let Y be a subspace of X . Then Y is Noetherian if and only if for each irreducible closed subset C of X , $Y \cap C = Y \cap C'$ for some closed subset $C' \supseteq C$ such that $X \setminus C'$ is quasicompact.*

In later sections we focus on spectral spaces X in which the set of maximal elements under the specialization order \leq of X is a Noetherian space. The spectral spaces in our applications have the additional property that (X, \leq) is a tree. In this case, as we show in Theorem 2.5, the Noetherian property for the maximal elements descends to subsets consisting of incomparable elements.

Lemma 2.4. *Let X be a spectral space whose specialization order \leq is a tree. Suppose that $\text{Max } X$ is a Noetherian space. Then a subspace Y of X is Noetherian if and only if (Y, \leq) satisfies the ascending chain condition.*

Proof. If Y is Noetherian, then the closed subsets of Y satisfy the descending chain condition, so (Y, \leq) satisfies the ascending chain condition. Conversely, suppose that (Y, \leq) satisfies ACC. Let C be an irreducible closed subset of X . By Lemma 2.3, to prove that Y is Noetherian, it suffices to show that there exists a closed subset $C' \supseteq C$ such that $Y \cap C = Y \cap C'$ and $X \setminus C'$ is quasicompact. By Lemma 2.3, there exists a closed subset $C_1 \supseteq C$ such that $C \cap \text{Max } X = C_1 \cap \text{Max } X$ and $X \setminus C_1$ is quasicompact. Since C is irreducible, there is $c \in C$ such that $C = \{x \in X : c \leq x\}$. Let $D = \bigcap_{y < c, y \in Y} \overline{\{y\}}$. Since (X, \leq) is a tree and (Y, \leq) satisfies ACC, C is a proper subset of D . Thus since the quasicompact open subsets of X form a basis for X , there is a closed set C_2 such that $C \subseteq C_2$, $D \not\subseteq C_2$ and $X \setminus C_2$ is quasicompact. We claim that $Y \cap C = Y \cap C_1 \cap C_2$. The containment " \subseteq " is clear since $C \subseteq C_1 \cap C_2$. Let $y \in Y \cap C_1 \cap C_2$. Then there exists $m \in \text{Max } X$ such that $y \leq m$. Thus $m \in \text{Max } X \cap C_1 \cap C_2 \subseteq C$, so that $c \leq m$. Since (X, \leq) is a tree and $y, c \leq m$, it must be that $y < c$ or $c \leq y$. If $y < c$, then $C \subsetneq D \subseteq \overline{\{y\}}$. However, since $y \in C_2$, this forces $D \subseteq C_2$, a contradiction. Thus $c \leq y$, and hence $y \in C$. This shows that $Y \cap C = Y \cap C_1 \cap C_2$. Finally, since $X \setminus C_1$ and $X \setminus C_2$ are quasicompact, so is their union $X \setminus (C_1 \cap C_2)$. Thus with $C' = C_1 \cap C_2$ the claim is proved. \square

Theorem 2.5. *Let X be a spectral space whose specialization order \leq is a tree. Then $\text{Max } X$ is a Noetherian space in the spectral topology if and only if every subset of*

X consisting of elements that are incomparable under \leq is discrete in the inverse topology.

Proof. Suppose $\text{Max} X$ is a Noetherian space. Let Y be a nonempty subset of X whose elements are incomparable under \leq . Then by Lemma 2.4, Y is a Noetherian space. Let $y \in Y$. Since the elements of Y are incomparable, $Y \setminus \{y\}$ is open in Y . In a Noetherian space, open sets are quasicompact, so $Y \setminus \{y\}$ is inverse closed in Y , which proves that y is isolated in the inverse topology on Y .

Conversely, suppose that every subspace of X consisting of incomparable elements is discrete in the inverse topology. To prove that $\text{Max} X$ is Noetherian, it suffices by Lemma 2.3 to show that for each irreducible closed subset C of X there exists a closed set $C' \supseteq C$ such that $C \cap \text{Max} X = C' \cap \text{Max} X$ and $X \setminus C'$ is quasicompact. Let C be an irreducible closed subset of X , and let $c \in C$ such that $C = \{x \in X : c \leq x\}$. By assumption, $\{c\} \cup ((\text{Max} X) \setminus C)$ is discrete in the inverse topology since the elements in this set are incomparable. Thus there exists a quasicompact open subset U of X such that $(\text{Max} X) \setminus C \subseteq U$ and $c \notin U$. Since $c \notin U$ and U , being open, has the property that $U = \downarrow U$, it must be that $C \cap U = \emptyset$. Thus $U \subseteq X \setminus C$, so that $(\text{Max} X) \cap U \subseteq (\text{Max} X) \setminus C$. Since also $(\text{Max} X) \setminus C \subseteq U$, we conclude $(\text{Max} X) \setminus C = (\text{Max} X) \cap U$. Thus with $C' = X \setminus U$, we have $(\text{Max} X) \cap C = (\text{Max} X) \cap C'$ and $X \setminus C'$ is quasicompact. Since also $C \subseteq X \setminus U = C'$, the claim is proved. \square

Corollary 2.6. *Let X be a spectral space whose specialization order \leq is a tree. If $\text{Max} X$ is a Noetherian space, then $\text{Min} C$ is finite for each nonempty closed subset C of X .*

Proof. Let C be a nonempty closed subset of X . By Proposition 2.1(1), $\text{Min} C$ consists of the elements that are maximal with respect to the specialization order in the inverse topology. The set of maximal elements of a spectral space is a quasicompact subspace, so $\text{Min} C$ is quasicompact in the inverse topology. Thus since by Theorem 2.5, $\text{Min} C$ is discrete in the inverse topology, $\text{Min} C$ is finite. \square

3 Spectral representations

Throughout this section A, C and D are nonempty sets with $A \subsetneq C \subseteq D$. We do not assume the presence of any algebraic structure on these sets. We work under the assumption that A can be represented as an intersection of C with sets B between A and D such that the sets B are points in a spectral space X whose specialization order is compatible with set inclusion. The set C can be viewed as a fixed component in the intersection (e.g., the case $C = D$ is often an interesting choice). The goal then is to make “efficient” choices from X to represent A . We formalize some of this with the following definition.

Definition 3.1. Let X be a collection of subsets of D , and assume that X is a C -representation of A , meaning that $A = (\bigcap_{B \in X} B) \cap C$.

- (1) For each $F \subseteq D$, let $\mathcal{V}(F) = \{B \in X : F \subseteq B\}$ and $\mathcal{U}(F) = \{B \in X : F \not\subseteq B\}$. We say the C -representation X is *spectral* if X is a spectral space and $\{\mathcal{U}(d) : d \in D\}$ is a subbasis for X consisting of quasicompact open sets. Note that this choice of subbasis assures that the specialization order agrees with the partial order given by set inclusion.

Now assume that X is a spectral C -representation of A .

- (2) Let $Z \subseteq X$ be a C -representation of A , and let $B \in Z$. Then B is *irredundant* in Z if $Z \setminus \{B\}$ is not a C -representation of A ; B is *strongly irredundant* in Z if the only closed subset Y of $\mathcal{V}(B)$ such that $(Z \setminus \{B\}) \cup Y$ is a C -representation of A is $Y = \mathcal{V}(B)$; B is *tightly irredundant* in Z if $(Z \cup \mathcal{V}(B)) \setminus \{B\}$ is not a C -representation of A .
- (3) A closed subset Y of X (resp., a patch) that is minimal with respect to set inclusion among closed (resp., patch) C -representations of A in X is a *minimal closed (resp., patch) C -representation of A in X* .
- (4) A subspace of X of the form $\text{Min} Y$ for some minimal closed C -representation Y of A is a *minimal C -representation of A* .

The notions of strong and tight irredundance become much clearer in the settings of Sections 4-7; see the discussion after Definition 4.1.

Observe that in (4), since the specialization order on X agrees with the partial order given by set inclusion among the members of X , $\text{Min} Y$ is also the minimal elements of Y with respect to set inclusion.

Lemma 3.2. *Every spectral C -representation of A contains a minimal closed C -representation of A and a minimal patch C -representation of A .*

Proof. Let X be a spectral C -representation of A , and let \mathcal{F} be the set of closed C -representations of A in X . Then \mathcal{F} is nonempty since $X \in \mathcal{F}$. Let $\{Y_\alpha\}$ be a chain of elements in \mathcal{F} , and let $Y = \bigcap_\alpha Y_\alpha$. As an intersection of closed subsets, Y is closed. We claim that Y is a C -representation of A . Clearly, $A \subseteq (\bigcap_{B \in Y} B) \cap C$. Let $d \in (\bigcap_{B \in Y} B) \cap C$. Then $\bigcap_\alpha Y_\alpha = Y \subseteq \mathcal{V}(d)$, and hence $\mathcal{U}(d) \subseteq \bigcup_\alpha Y_\alpha^c$, where $Y_\alpha^c = X \setminus Y_\alpha$. Since $\mathcal{U}(d)$ is quasicompact and each Y_α^c is open, the fact that the Y_α form a chain under inclusion implies that $\mathcal{U}(d) \subseteq Y_\alpha^c$ for some α . For this choice of α , $Y_\alpha \subseteq \mathcal{V}(d)$, which since Y_α is a C -representation of A implies that $d \in (\bigcap_{B \in Y_\alpha} B) \cap C = A$. Therefore, $A = (\bigcap_{B \in Y} B) \cap C$, which shows that $Y \in \mathcal{F}$. By Zorn's Lemma, \mathcal{F} contains minimal elements. Since the patch topology is spectral by Proposition 2.1(2), the final statement follows from the first. \square

Lemma 3.3. *Let X be a spectral C -representation of A , let $Z \subseteq X$ be a C -representation of A and let $B \in Z$. Then*

- (1) B is irredundant in Z if and only if B is irredundant in \tilde{Z} .
- (2) B is tightly irredundant in Z if and only if B is irredundant in \bar{Z} .

Moreover, if B is irredundant in Z , then B is isolated in the spectral and patch subspace topologies on Z .

Proof. (1) Suppose that B is irredundant in Z . Then there exists $d \in D$ such that $d \notin B$ but d is in every other set that is in Z . Thus $Z \subseteq \mathcal{V}(d) \cup \{B\}$, and since $\mathcal{V}(d)$ and $\{B\}$ are patches in X , we have $\tilde{Z} \subseteq \mathcal{V}(d) \cup \{B\}$. Hence $\tilde{Z} \setminus \{B\} \subseteq \mathcal{V}(d)$. Since $d \notin B$, this implies that B is irredundant in \tilde{Z} . The converse is clear since $Z \subseteq \tilde{Z}$.

(2) Suppose that V is tightly irredundant in Z . Then there exists $d \in D$ such that d is not in B but d is in every any other set in $Z \cup \mathcal{V}(B)$. Thus $(Z \cup \mathcal{V}(B)) \setminus \{B\} \subseteq \mathcal{V}(d)$, which implies that $\mathcal{V}(d) \cup \mathcal{V}(B) = \mathcal{V}(d) \cup \{B\}$. Since $\mathcal{V}(d) \cup \mathcal{V}(B)$ is closed, we have $\bar{Z} \subseteq \mathcal{V}(d) \cup \mathcal{V}(B) = \mathcal{V}(d) \cup \{B\}$. Therefore, $\bar{Z} \setminus \{B\} \subseteq \mathcal{V}(d)$, and hence d is in every set in $\bar{Z} \setminus \{B\}$. Since $d \notin B$, we conclude that B is irredundant in \bar{Z} . Conversely, if B is irredundant in \bar{Z} , then since $\mathcal{V}(B) \subseteq \bar{Z}$, it follows that B is tightly irredundant in Z .

It remains to prove the last statement of the lemma. Suppose that B is irredundant in Z . Let $Z' = Z \setminus \{B\}$. Then there exists $c \in (\bigcap_{B' \in Z'} B') \cap C$ with $c \notin B$. Since the set $\mathcal{V}(c)$ is closed in the spectral and patch topologies and this set contains Z' but not B , we have that B is an isolated point in the spectral and patch subspace topologies on Z . \square

Proposition 3.4. *Let X be a spectral C -representation of A . If $Z \subseteq X$ is a tightly irredundant C -representation of A , then Z is contained in a minimal C -representation of A . Thus the number of minimal C -representations of A is greater than the number of tightly irredundant C -representations of A .*

Proof. By Lemma 3.3, the members of Z are irredundant in the C -representation \bar{Z} of A . By Lemma 3.2, there exists a minimal C -representation Z_1 contained in \bar{Z} . Since the members of Z are irredundant in \bar{Z} , it must be that $Z \subseteq Z_1$. To prove the last claim of the proposition, it suffices to show that distinct irredundant C -representations of A are contained in distinct minimal C -representations of A . Suppose $Y \subseteq X$ is another tightly irredundant C -representation of A with $Y \neq Z$. Then the members of Y are irredundant in a minimal C -representation Y_1 of A . If $Y_1 = Z_1$, then the members of Y and Z are irredundant in Y_1 , which, since Y and Z are C -representations of A , implies that $Y = Z$, a contradiction that implies $Y_1 \neq Z_1$. \square

Example 4.3 shows that neither tightly irredundant nor minimal representations need be unique.

Lemma 3.5. *Let X be a spectral C -representation of A , and let Z be a minimal C -representation of A in X . Then \bar{Z} is a minimal closed C -representation of A , \tilde{Z} is a minimal patch C -representation of A , $Z = \text{Min } \tilde{Z} = \text{Min } \bar{Z}$ and $\bar{Z} = \uparrow Z$.*

Proof. Since Z is a minimal C -representation, there exists a minimal closed C -representation Y of A such that $Z = \text{Min } Y$. Since $Z \subseteq Y$ and Y is closed, $\bar{Z} \subseteq Y$. Thus the minimality of Y forces $\bar{Z} = Y$, and hence \bar{Z} is a minimal closed C -representation of A . Also, note that this implies that $\text{Min } \bar{Z} = \text{Min } Y = Z$.

Suppose that Y is a patch C -representation of A with $Y \subseteq \tilde{Z}$. We claim that $Y = \tilde{Z}$, and we prove this by first showing that $Z \subseteq Y$. By Proposition 2.1(3), $\uparrow Y$ is a closed C -representation of A . Also by Proposition 2.1(3), $\uparrow Y \subseteq \uparrow(\tilde{Z}) = \bar{Z}$, so that the minimality of \bar{Z} forces $\uparrow Y = \bar{Z}$. Since \bar{Z} is closed, $\uparrow(\text{Min } \bar{Z}) = \bar{Z} = \uparrow Y$. We have

established that $Z = \text{Min } \bar{Z}$, so $\uparrow Y = \uparrow Z$. Now $\text{Min } \uparrow Y = \text{Min } Y$, and since Z consists of pairwise incomparable elements, $\text{Min } \uparrow Z = Z$. Thus $\text{Min } Y = Z$, so that $Z \subseteq Y$. Since Y is a patch, $\tilde{Z} \subseteq Y$. Therefore, $Y = \tilde{Z}$. Since $\text{Min } Y = Z$, we conclude from $\tilde{Z} = Y$ that $\text{Min } \tilde{Z} = \text{Min } Y = Z$. All that remains is to show that $\bar{Z} = \uparrow Z$. Since $\text{Min } \bar{Z} = Z$, it follows that $\bar{Z} = \uparrow \text{Min } \bar{Z} = \uparrow Z$. \square

We obtain now a topological characterization of irredundance in minimal C -representations. Example 4.6 shows that without the restriction to minimal representations, irredundance may not have a similar topological expression.

Theorem 3.6. *Let X be a spectral C -representation of A , and let Z be a minimal C -representation of A in X . Then the spectral and patch subspace topologies agree on Z , and the following are equivalent for $B \in Z$.*

- (1) B is irredundant in Z .
- (2) B is strongly irredundant in Z .
- (3) B is isolated in the spectral (equivalently, patch) subspace topology on Z

Proof. By assumption, there exists a minimal closed C -representation Y of A such that $Z = \text{Min } Y$. Since Y is a patch in X , we have by Proposition 2.1(4)(a) that Y is spectral in the spectral subspace topology. Also by Proposition 2.1(4)(c), the spectral topology on the minimal points of a spectral space is the same as the patch topology. Thus the spectral and patch topologies agree on Z .

That (2) implies (1) is clear, and that (1) implies (3) follows from Lemma 3.3. It remains to prove that (3) implies (2). Suppose B is isolated in Z , so that $B \notin \overline{Z \setminus \{B\}}$. To see that B is strongly irredundant in Z , let F be a closed subset of $\mathcal{V}(B)$ such that $(Z \setminus \{B\}) \cup F$ is a C -representation of A . Now $F \subseteq \mathcal{V}(B) \subseteq \bar{Z}$, so $Z' := (\overline{Z \setminus \{B\}}) \cup F$ is a closed C -representation of A contained in \bar{Z} . By Lemma 3.5, \bar{Z} is a minimal closed C -representation of A , so $Z' = \bar{Z}$, and hence $B \in Z'$. Since $B \notin \overline{Z \setminus \{B\}}$, we conclude that $B \in F$ and hence $F = \mathcal{V}(B)$. Thus B is strongly irredundant in Z . \square

Corollary 3.7. *Let X be a spectral C -representation of A , and let Z be a minimal C -representation of A in X . Then Z contains a (strongly) irredundant C -representation of A if and only if the set of isolated points in Z is dense in Z . Hence there is at most one irredundant C -representation of A in \tilde{Z} .*

Proof. By Theorem 3.6 the patch and spectral topologies agree on Z , so in the proof we work exclusively in the patch topology. Suppose Z is a minimal C -representation of A in X that contains an irredundant C -representation Y of A . Since Y is a C -representation of A , so is \tilde{Y} . Thus since by Lemma 3.5, \tilde{Z} is a minimal patch C -representation of A , we have $\tilde{Y} = \tilde{Z}$. Therefore, Y is dense in Z . Moreover, by Lemma 3.3, each member of Y is irredundant in $\tilde{Y} = \tilde{Z}$, hence in Z . Therefore, by Theorem 3.6, each member of Y is isolated in Z .

Conversely, suppose that the set Y of isolated points in Z is dense in Z . If Y is not a C -representation of A , then there exists $B' \in Z$ and $c \in (\bigcap_{B \in Y} B) \cap C$ with $c \notin B'$. Thus $Y \subseteq \mathcal{V}(c)$ and $B' \notin \mathcal{V}(c)$, so that $\mathcal{U}(c) \cap Y = \emptyset$ while $\mathcal{U}(c) \cap Z \neq \emptyset$, a contradiction to the assumption that Y is dense in Z . Therefore, Y is a C -representation of A ,

and hence by Theorem 3.6, the members of Y are strongly irredundant in the C -representation Z .

To prove the last claim of the corollary, suppose Y is an irredundant C -representation of A in \tilde{Z} . By Lemma 3.3, the elements of Y are isolated points in \tilde{Z} with respect to the patch topology. Thus for each $y \in Y$, $\{y\}$ is open in \tilde{Z} , so that since Z is dense in \tilde{Z} , we must have $y \in Z$. Therefore, $Y \subseteq Z$. Since Y is an irredundant C -representation of A , Theorem 3.6 implies that Y is the set of isolated points of Z . Hence there is at most one irredundant C -representation of A in \tilde{Z} , namely the set of isolated points of Z . \square

A topological space X is *scattered* if every nonempty subspace Y of X contains a point that is isolated in Y ; equivalently, in every nonempty subset Y of X the set of isolated points in Y is dense in Y .

Corollary 3.8. *Let X be a spectral C -representation of A . If X is scattered in the spectral or patch topologies, then X contains a strongly irredundant C -representation of A .*

Proof. Since the patch topology refines the spectral topology, to be scattered in the spectral topology implies the space is scattered in the patch topology. Thus we assume that X is scattered in the patch topology. Let Z be a minimal C -representation of A . Since X is scattered, the set of isolated points in Z is dense in Z with respect to the patch topology, so by Corollary 3.7, Z contains a strongly irredundant C -representation of A . \square

Corollary 3.9. *If X is a countable spectral C -representation of A , then X contains a strongly irredundant C -representation of A .*

Proof. Since X is spectral, the patch topology on X is compact and Hausdorff. A countable compact Hausdorff space is homeomorphic to an ordinal space [39], and hence scattered since an ordinal space is well-ordered. Thus X is scattered in the patch topology, and by Corollary 3.8, X contains a strongly irredundant C -representation of A . \square

We single out next the members of X that must appear in every closed C -representation of A . These members play an important role in the applications to intersections of valuation rings in Sections 5 and 6.

Definition 3.10. Let X be a spectral C -representation of A . Let Y be the intersection of all closed C -representations of A in X (so that Y is a closed set in X , but not necessarily a C -representation of A). An element $B \in X$ is *critical* for the C -representation X if $B \in Y$. Since Y is a closed subset of the spectral space X , Y contains minimal elements. We define $\mathcal{C}(X) = \text{Min } Y$.

Proposition 3.11. *Let X be a spectral C -representation of A . Then B is critical in X if and only if whenever $A = A_1 \cap \cdots \cap A_n \cap C$, where each A_i is an intersection of members of X , it must be that $A_i \subseteq B$ for some i .*

Proof. Suppose B is critical in X and $A = A_1 \cap \cdots \cap A_n \cap C$, where each A_i is an intersection of members of X . Then $\mathcal{V}(A_1) \cup \cdots \cup \mathcal{V}(A_n)$ is a closed C -representation of A in X . Since B is critical in X , $B \in \mathcal{V}(A_i)$ for some i , and hence $A_i \subseteq B$. Conversely, suppose that whenever $A = A_1 \cap \cdots \cap A_n \cap C$, where each A_i is an intersection of members of X , we have $A_i \subseteq B$ for some i . Let Y be a closed C -representation of A in X . Since Y is closed and X is spectral, Y is an intersection of sets of the form $\mathcal{V}(F_1) \cup \cdots \cup \mathcal{V}(F_n)$, where each F_i is a finite subset of C . Thus to show that $B \in Y$ it suffices to show B is in every set of this form that contains Y . Let F_1, \dots, F_n be finite subsets of C such that $Y \subseteq \mathcal{V}(F_1) \cup \cdots \cup \mathcal{V}(F_n)$. For each i , let $A_i = \bigcap_{E \in \mathcal{V}(F_i)} E$. Since Y is a C -representation of A , we have $A = A_1 \cap \cdots \cap A_n \cap C$. By assumption, $A_i \subseteq B$ for some i , so $F_i \subseteq B$. Thus $B \in \mathcal{V}(F_i)$, which proves the proposition. \square

The next corollary shows that for critical members of X , being irredundant in a representation is the same as being strongly irredundant.

Corollary 3.12. *Let X be a spectral C -representation of A . If $B \in X$ is critical in X and B is irredundant in some C -representation Z of A in X , then B is strongly irredundant in Z .*

Proof. Suppose there is a nonempty closed subset Y of $\mathcal{V}(B)$ such that $(Z \setminus \{B\}) \cup Y$ is a C -representation of A . We claim that $B \in Y$. Since Y is closed, Y is an intersection of sets of the form $\mathcal{V}(F_1) \cup \cdots \cup \mathcal{V}(F_n)$, where F_1, \dots, F_n are finite subsets of D . Let F_1, \dots, F_n be finite subsets of D such that $Y \subseteq \mathcal{V}(F_1) \cup \cdots \cup \mathcal{V}(F_n)$. Then

$$A = \left(\bigcap_{B' \in \mathcal{V}(F_1)} B' \right) \cap \cdots \cap \left(\bigcap_{B' \in \mathcal{V}(F_n)} B' \right) \cap \left(\bigcap_{B' \in Z \setminus \{B\}} B' \right) \cap C \subseteq B.$$

Since B is irredundant in Z , we have $\bigcap_{B' \in Z \setminus \{B\}} B' \not\subseteq B$. Thus since B is critical in X , we conclude that $F_i \subseteq \bigcap_{B' \in \mathcal{V}(F_i)} B' \subseteq B$ for some i . Hence $B \in \mathcal{V}(F_1) \cup \cdots \cup \mathcal{V}(F_n)$, which shows that $B \in Y$. \square

Next we prove a uniqueness theorem for strongly irredundant C -representations when $\mathcal{C}(X)$ has enough members to be itself a C -representation of A . This case is important in Section 6, where we work with v -domains, a class of rings that can be represented as an intersection of their critical valuation overrings.

Theorem 3.13. *Let X be a spectral C -representation of A . Then $A = (\bigcap_{B \in \mathcal{C}(X)} B) \cap C$ if and only if A has a unique minimal C -representation in X . If this is the case, then the following statements hold for the set*

$$S = \{B \in X : B \text{ is strongly irredundant in some } C\text{-representation of } A \text{ in } X\}.$$

- (1) $S \subseteq \mathcal{C}(X)$ and hence every member of S is critical in X .
- (2) Each $B \in S$ is strongly irredundant in the C -representation $\mathcal{C}(X)$.
- (3) If A has a strongly irredundant C -representation Z in X , then $Z = S$.
- (4) X contains at most one strongly irredundant C -representation of A .

Proof. Observe first that $\uparrow\mathcal{C}(X)$ is the intersection of all the closed C -representations of A in X . Thus $A = (\bigcap_{B \in \mathcal{C}(X)} B) \cap C$ if and only if $\uparrow\mathcal{C}(X)$ is a C -representation of A , if and only if there is a unique minimal closed C -representation of A , if and only if there is a unique minimal C -representation of A in X .

(1) Let $B \in S$. Then there is $Y \subseteq X$ such that $Y \cup \{B\}$ is a C -representation of A and B is strongly irredundant in $Y \cup \{B\}$. For $E \in \mathcal{C}(X)$, Proposition 3.11 implies $B \subseteq E$ or $\bigcap_{B' \in Y} B' \subseteq E$. If $\bigcap_{B' \in Y} B' \subseteq E$ for every $E \in \mathcal{C}(X)$, then since by assumption $\mathcal{C}(X)$ is a C -representation of A , this forces $A = (\bigcap_{B' \in Y} B') \cap C$, contrary to the irredundance of B in $\{B\} \cup Y$. Therefore, $Z := (\uparrow\mathcal{C}(X)) \cap \mathcal{V}(B)$ is nonempty, and for every $E \in \mathcal{C}(X)$ with $B \not\subseteq E$, it must be that $\bigcap_{B' \in Y} B' \subseteq E$. Thus since $\mathcal{C}(X)$ is a C -representation of A , so is $Z \cup Y$. Since $\uparrow\mathcal{C}(X)$ and $\mathcal{V}(B)$ are closed subsets of X , so is Z . Now $Z \subseteq \mathcal{V}(B)$, so since B is strongly irredundant in the C -representation $\{B\} \cup Y$, it must be that $Z = \mathcal{V}(B)$. Thus $B \in \uparrow\mathcal{C}(X)$.

Next we show that $B \in \mathcal{C}(X)$. There exists $E \in \mathcal{C}(X)$ such that $E \subseteq B$. Since $A = B \cap (\bigcap_{B' \in Y} B') \cap C$ and E is critical, Proposition 3.11 implies that $B = E$ or $\bigcap_{B' \in Y} B' \subseteq E$. Since $E \subseteq B$ and B is irredundant in $\{B\} \cup Y$, the latter cannot occur, so $B = E$. Therefore, B is minimal in $\mathcal{C}(X)$.

(2) As in (1), every $E \in \mathcal{C}(X)$ with $E \neq B$ contains $\bigcap_{B' \in Y} B'$, so that if B is not irredundant in $\mathcal{C}(X)$, then since $\mathcal{C}(X)$ is a C -representation of A , we have $A = (\bigcap_{B' \in Y} B') \cap C$, a contradiction. Thus B is irredundant in $\mathcal{C}(X)$, and by Corollary 3.12, B is strongly irredundant in $\mathcal{C}(X)$.

(3) Suppose Z is a strongly irredundant C -representation in X . By (1), $Z \subseteq \mathcal{C}(X)$ and the members of Z are strongly irredundant in $\mathcal{C}(X)$. Also by (2), the members of S are strongly irredundant in $\mathcal{C}(X)$. It follows that $S = Z$.

(4) This is clear from (3). \square

4 Irredundance in intersections of valuation rings

In this section, we reinterpret the material of Section 3 for the Zariski-Riemann space of a field. We assume the following notation throughout this section.

- A is a proper integrally closed subring of a field F .
- C is a set (not necessarily a ring) such that $A \subsetneq C \subseteq F$.
- \mathfrak{X} denotes the set of valuation rings of F containing A .

Zariski introduced a topology on \mathfrak{X} (the *Zariski topology*) by designating as a basis of open sets the sets of the form $\{V \in \mathfrak{X} : x_1, \dots, x_n \in V\}$, where $x_1, \dots, x_n \in F$. With this topology, the same topology as in Example 2.2(8), \mathfrak{X} is a spectral space and is termed the *Zariski-Riemann space* of F/A . For some recent articles emphasizing a topological approach to the Zariski-Riemann space, see [8, 9, 10, 11, 12, 13, 42, 44, 45].

Comparison of the basic opens in the Zariski topology on \mathfrak{X} with the topology in Section 3 shows that it is the inverse topology on \mathfrak{X} rather than the Zariski topology that is needed in order to deal with issues of irredundance. The inverse topology is

also a natural one to consider here since under this topology the specialization order on \mathfrak{X} agrees with the usual order on \mathfrak{X} given by set inclusion. To avoid confusing the two topologies, we denote by \mathfrak{X}^{-1} the set \mathfrak{X} with the inverse topology. Then \mathfrak{X}^{-1} is a spectral C -representation of A , and so all the results of Section 3 can be translated into the context of the Zariski-Riemann space of F by working inside the spectral C -representation \mathfrak{X}^{-1} .

In the spirit of Section 3, we work throughout this section relative to the set C and consider C -representations X of A ; that is, $A = (\bigcap_{V \in X} V) \cap C$, where X is a collection of valuation rings in \mathfrak{X} . That C need only be a subset in most cases is a byproduct of the approach in Section 2. While we do not have an application for the level of generality that working with a set rather than a ring affords, we do so anyway since it comes at no extra expense. When $C = F$, we abbreviate “ C -representation” to “representation.” Thus a *representation* of A is a subset X of \mathfrak{X} such that $A = \bigcap_{V \in X} V$.

In this section \mathfrak{X}^{-1} will play the role that X did in Section 3 of an ambient spectral representation. In this section we use “ X ” then for not necessarily spectral subsets of \mathfrak{X} . A C -representation X of A then is a subspace of \mathfrak{X}^{-1} . In particular:

When applying the results of Section 3, the default topology on the C -representations of A is the inverse topology. Thus the specialization order coincides with set inclusion among the valuation rings, and the operators $\text{Min}(-)$ and $\text{Max}(-)$ yield the minimal and maximal elements, respectively, of a collection of valuation rings with respect to set inclusion.

Definition 4.1. Let $X \subseteq \mathfrak{X}$. We define $\text{cl}(X)$, $\text{inv}(X)$ and $\text{patch}(X)$ to be the closure of X in the Zariski, inverse and patch topologies, respectively. We denote by $\text{gen}(X)$ the set of generalizations of the valuation rings in X ; that is,

$$\text{gen}(X) = \{V \in \mathfrak{X} : W \subseteq V \text{ for some } W \in X\}.$$

We interpret now the results of Section 3 in the setting of the Zariski-Riemann space. The notions of irredundance from Definition 3.1(2) can be simplified for valuation rings. Let X be a subset of \mathfrak{X} such that $A = (\bigcap_{V \in X} V) \cap C$. Then $V \in X$ is *irredundant* in the C -representation X if V cannot be omitted from this intersection; V is *strongly irredundant* if V cannot be replaced in this intersection by a valuation overring¹; and V is *tightly irredundant* if V cannot be replaced by an intersection of valuation overrings that properly contain V .

Combining Lemmas 3.2 and 3.5, and observing that in the notation of Section 2, $\text{gen}(X) = \downarrow X$, we have the following existence result for minimal representations.

(4.2) *In every inverse closed subset X of \mathfrak{X} there is a minimal C -representation of A ; that is, there exists in X a collection Z of pairwise incomparable valuation rings such that $\text{gen}(Z)$ is a minimal inverse closed C -representation of A and $\text{patch}(Z)$ is a minimal patch C -representation of A .*

In general, there can exist infinitely many such minimal C -representations of A . This is illustrated by Example 4.3.

¹ By an *overring* of a domain R we mean a ring between R and its quotient field.

Example 4.3. In [41, Example 6.2], an integrally closed overring A of $K[X, Y, Z]$, with K any field and X, Y, Z indeterminates, is constructed such that A has uncountably many strongly irredundant representations. Since every valuation overring of $K[X, Y, Z]$ has finite Krull dimension, a valuation ring in a representation of A is strongly irredundant if and only if it is tightly irredundant (see the discussion after (4.4)). Thus by Proposition 3.4, A has uncountably many minimal representations.

Applying Lemma 3.3, we have

(4.4) *If $A = (\bigcap_{V \in X} V) \cap C$, then $V \in X$ is irredundant in X if and only if V is irredundant in $\text{patch}(X)$; V is tightly irredundant in X if and only if V is irredundant in $\text{inv}(X)$.*

If V has finite Krull dimension, then since there are only finitely many overrings of V , V is strongly irredundant in the C -representation X if and only if V is tightly irredundant in X . More generally, if the maximal ideal of V is not the union of the prime ideals properly contained in it, then the notions of strong and tight irredundance coincide for V . In particular, if $V \in X$ has rank one, then V is irredundant in the C -representation X if and only if V is irredundant in $\text{inv}(X)$.

While an irredundant member V of a C -representation X is by Lemma 3.3 an isolated point in the inverse and patch topologies on Z , the converse need not be true, as illustrated by Example 4.6. However, by restricting to minimal C -representations we obtain from Theorem 3.6 that irredundance is topological for such representations.

(4.5) *Suppose X is a minimal C -representation of A , as in (4.2). A valuation ring $V \in X$ is irredundant in X if and only if V is strongly irredundant in X ; if and only if V is isolated in X in the inverse (equivalently, patch) topology.*

Example 4.6. A valuation ring V in a C -representation X of A may be isolated in the inverse topology on X but be redundant in X . For example, let A be an integrally closed Noetherian local domain of Krull dimension > 1 , let $X = \{A_P : P \text{ is a height one prime ideal of } A\}$, and let V be a DVR overring of A that dominates A . Write the maximal ideal M of A as $M = (a_1, \dots, a_n)$. Then X is a subset of the inverse closed set $Y := \{W \in \mathfrak{X} : 1/a_i \in W \text{ for some } i = 1, \dots, n\}$, while $V \notin Y$. Thus V is an isolated point in $\{V\} \cup X$ with respect to the inverse topology. However, since $A = \bigcap_{W \in X} W$, V is redundant in the representation $\{V\} \cup X$ of A . (The notion of a minimal representation remedies this: X is a minimal representation so that V is excluded from consideration since it is not an element of X .)

By Corollary 3.7, the existence of a strongly irredundant C -representation within a minimal C -representation depends only on the topology of the minimal representation:

(4.7) *Suppose X is a minimal C -representation of A , as in (4.2). Then X contains a strongly irredundant C -representation Y of A if and only if the set of isolated points in X is dense in X with respect to the inverse topology.*

In such a case the only choice for Y is the set of isolated points of X , and hence there exists at most one such irredundant C -representation of A in X , hence also in

$\text{patch}(X)$ (Lemma 3.3). However, moving outside of $\text{patch}(X)$, Example 4.3 shows there can exist infinitely many distinct strongly irredundant C -representations of A . This example involves an intersection of valuation overrings of a three-dimensional Noetherian domain. By contrast, strongly irredundant representations over two-dimensional Noetherian domains are much better behaved and have a number of uniqueness properties [41].

One consequence of the topological approach of Section 3 is an existence result for strongly irredundant C -representations of A in the countable case. This result, which follows from Corollary 3.9, is revisited in the next section in Theorem 6.8.

(4.8) *If $A = (\bigcap_{V \in X} V) \cap C$ for some countable patch X in \mathfrak{X} , then X contains a strongly irredundant C -representation of A .*

It is important here that we work with a countable patch rather than simply a countable subset of \mathfrak{X} . This is illustrated by the next example.

Example 4.9. Suppose A is a countable integrally closed local Noetherian domain with maximal ideal M and quotient field F . Suppose also that A has Krull dimension > 1 . Let X be the collection of all DVR overrings V of A that are centered in A on M and such that V is a localization of the integral closure of some finitely generated A -subalgebra of F . Since A is countable and Noetherian, there are countably many such valuation rings. Moreover, A is the intersection of the valuation rings in X , since if $x \in F \setminus A$, then there exists a valuation ring V in X whose maximal ideal contains x^{-1} , so that $x \notin V$. If $V \in X$ is an irredundant representative of A , then since the value group of V is a subgroup of the group of rational numbers, V is a localization of A [29, Lemma 1.3], a contradiction to the fact that A has Krull dimension > 1 and V is centered on the maximal ideal of A . Therefore, although X is countable, X contains no irredundant representatives of A . It follows from (4.8) that X is not a patch closed subspace of \mathfrak{X} .

Adapting the terminology from Section 3, we say a valuation ring $V \in \mathfrak{X}$ is *C-critical for A* if V is an element of every inverse closed C -representation of A . Thus by Proposition 3.11 and the fact that every integrally closed A -subalgebra of F is an intersection of valuation rings in \mathfrak{X} , we have

(4.10) *V is C -critical for A if and only if whenever A_1, \dots, A_n are integrally closed A -subalgebras of F such that $A = A_1 \cap \dots \cap A_n \cap C$, it must be that $A_i \subseteq V$ for some i .*

Also, from Corollary 3.12 we see that if V is C -critical for A and irredundant in some C -representation X of A , then V is strongly irredundant in X . By restricting to the case where C is an A -submodule of F , we obtain an important class of C -critical valuation rings; these are the valuation rings that play an important role in the next sections. A valuation ring $V \in \mathfrak{X}$ is *essential for A* if $V = A_P$ for a prime ideal P of A .

Proposition 4.11. *Let $V \in \mathfrak{X}$ such that $C \not\subseteq V$. If C is an A -submodule of F and V is essential for A , then V is C -critical for A .*

Proof. We use Proposition 3.11 to prove the claim. Let P be a prime ideal of A such that $A_P = V$, and let A_1, \dots, A_n be integrally closed A -subalgebras of F such that $A = A_1 \cap \dots \cap A_n \cap C$. Then since localization commutes with finite intersections, we have $V = A_P = (A_1)_P \cap \dots \cap (A_n)_P \cap C_P$. Since V is a valuation ring, the set of V -submodules between V and F forms a chain. Therefore, since $C \not\subseteq V$, there is i such that $A_i \subseteq V$, and hence by Proposition 3.11, V is C -critical. \square

Example 4.12. A valuation overring that is C -critical for A need not be essential. Suppose A has quotient field F . Then A is said to be *vacant* if it has a unique Kronecker function ring [7]. (Kronecker function rings are discussed after 4.13.) As we see in (4.14), this implies that A has a unique minimal representation. Hence A is vacant if and only if every valuation overring of A is critical (see also [7]). As discussed in [7] there exist vacant domains that are not Prüfer domains, and hence such a domain has a critical valuation overring that is not essential.

Example 4.3 shows that in general A need not be an intersection of critical valuation overrings; equivalently, A need not have a unique minimal C -representation. However, for some well-studied classes of rings, such as those in the next two sections, it is possible to represent A with critical valuation rings. In this case, strong properties hold for A . For example, applying Theorem 3.13(1), we have the following fact.

(4.13) *Suppose $A = (\bigcap_{V \in \mathcal{C}(\mathfrak{X})} V) \cap C$, where $\mathcal{C}(\mathfrak{X})$ is the set of minimal C -critical valuation rings in \mathfrak{X} . If $V \in \mathfrak{X}$ is strongly irredundant in some C -representation of A , then V is in $\mathcal{C}(\mathfrak{X})$ and V is strongly irredundant in $\mathcal{C}(\mathfrak{X})$.*

Thus $\mathcal{C}(\mathfrak{X})$ collects all the strongly irredundant representatives of A , and so, as in Theorem 3.13, having a strongly irredundant representation is a matter of having enough strongly irredundant representatives.

(4.13) *Suppose $A = (\bigcap_{V \in \mathcal{C}(\mathfrak{X})} V) \cap C$, so that $\mathcal{C}(\mathfrak{X})$ is a C -representation of A . Then A has a strongly irredundant C -representation if and only if A is an intersection of C with valuation rings in the set*

$$\{V \in \mathfrak{X} : V \text{ is strongly irredundant in some } C\text{-representation of } A\}.$$

Thus A has at most one strongly irredundant C -representation.

There is a long tradition of using Kronecker function rings to represent integrally closed rings in the field F with a Bézout domain in a transcendental extension $F(T)$ of F . We depart from this tradition because of our emphasis on the more general topological approach via spectral representations as in Section 3. However, in the present context of Zariski-Riemann spaces there is a precise connection between minimal representations and maximal Kronecker function rings. In fact, minimal representations play for us a role similar to that played by the Kronecker function ring in articles such as [2, 3, 24, 42]. We outline this connection here.

Let T be an indeterminate for F . For each valuation ring $V \in \mathfrak{X}$, let $V^* = V[T]_{\mathfrak{M}_V[T]}$, where \mathfrak{M}_V is the maximal ideal of V . Then V^* is a valuation ring with quotient field $F(T)$ such that $V = V^* \cap F$. For a nonempty subset X of \mathfrak{X} , the *Kronecker function ring of X* is the ring

$$\mathrm{Kr}(X) = \bigcap_{V \in X} V^*.$$

Then $\mathrm{Kr}(X)$ is a Bézout domain with quotient field $F(T)$; cf. [9, Corollary 3.6], [28, Theorem 2.2] and [32, Corollary 2.2]. When X is a C -representation of A , then $A = \mathrm{Kr}(X) \cap C$, and we say that $\mathrm{Kr}(X)$ is a *Kronecker C -function ring of A* . Thus to every C -representation of A corresponds a Kronecker C -function ring of A .

For each $X \subseteq \mathfrak{X}$, let $X^* = \{V^* : V \in X\}$. The mapping $\mathfrak{X} \rightarrow \mathfrak{X}^*$ is a homeomorphism with respect to the Zariski topology (see [9, Corollary 3.6] or [32, Proposition 2.7]), and hence is a homeomorphism in the inverse and patch topologies also. The subset X is inverse closed in \mathfrak{X} if and only if X^* is the set of localizations at prime ideals of $\mathrm{Kr}(X)$; i.e., X^* is the Zariski-Riemann space of the Bézout domain $\mathrm{Kr}(X)$ [45, Proposition 5.6]. Moreover, we have the following connection between C -representations and Kronecker C -function rings, which can be deduced from [45, Corollary 5.8 and Proposition 5.10].

(4.14) *The inverse closed C -representations of A bijectively correspond to the Kronecker C -function rings of A . The minimal C -representations of A bijectively correspond to the maximal Kronecker C -function rings of A . Moreover, a subset X of \mathfrak{X} is a minimal C -representation of A if and only if X^* consists of the localizations at maximal ideals of a maximal Kronecker C -function ring of A .*

5 Generalizations of Krull domains

In this section we assume the same notation as Section 4. Thus A is an integrally closed subring of the field F , C is a set between A and F , and \mathfrak{X} is the Zariski-Riemann space of F/A . Intersection representations play an important role in the theory of *Krull domains*, those integral domains that can be represented by a finite character intersection of rank one discrete valuation rings (DVRs). (A subset X of \mathfrak{X} has *finite character* if each $0 \neq x \in F$ is a unit in all but at most finitely many valuation rings in X .) Finite character representations of a Krull domain A are well understood: The collection $X = \{A_P : P \text{ a height one prime ideal of } A\}$ is a finite character, irredundant representation of DVRs. Krull [35] proved more generally that if A has a finite character representation consisting of valuation rings whose value groups have rational rank one, then this collection can be refined to one in which every valuation ring is essential for A ; see also [40, Corollary 5.2]. Examples due to Griffin [26, Section 4], Heinzer and Ohm [30, 2.4] and Ohm [40, Example 5.3] show that the same is not true if the value groups of the valuation rings are assumed only to have rank one rather than rational rank one.

Griffin defines the ring A to have *Krull type* if A has a finite character representation X consisting of essential valuation rings [26, 27]. In [26, Theorem 7] he gives necessary conditions for a ring A having a finite character representation of valuation rings to be a ring of Krull type. Pirtle [47, Corollary 2.5] showed that when

in addition the valuations in X have rank one, X is an irredundant representation of A . More generally, Brewer and Mott [3, Theorem 14] prove that if A has a finite character representation X of valuation rings (no restriction on rank), then A has an irredundant finite character representation, and if also the valuation rings in X have rank one, then A has one, and only one, irredundant finite character representation consisting of rank one valuation rings [3]. In [2, Theorem 1.1], Brewer proves that if A has Krull type, then A has an irredundant finite character representation X consisting of essential valuation rings, and that X is unique among such representations.

In both the articles [2] and [3], the authors prove their results by passing to a maximal Kronecker function ring of A and applying Gilmer and Heinzer's theory of irredundant representations of Prüfer domains to work out the problem of irredundance in a Prüfer setting. This method of passage to a Kronecker function ring, and hence to a Prüfer domain, is applied in [44] to domains A that can be represented with a collection of valuation rings from a Noetherian subspace of the Zariski-Riemann space, a class of representations that subsumes the finite character ones. In such a case A can be represented by a strongly irredundant Noetherian space of valuation rings [44, Theorem 4.3]. The results in [44] are in fact framed in terms of C -representations, where C is a ring.²

The introduction of finite character rank one C -representations to generalize the theory of Krull domains is due to Heinzer and Ohm [30]. This allows for considerable more flexibility in applying results to settings in which one considers, say, integrally closed rings between A and some integrally closed overring C . Even when A is a two-dimensional Noetherian domain and C is chosen a PID, the analysis of the integrally closed rings between A and C is quite subtle; see for example [1, 6, 38, 42, 46]. Regardless of the choice of A and C , Heinzer and Ohm [30, Corollary 1.4] prove that finite character rank one C -representations remain as well behaved as in the classical case of $C = F$: If C is a ring and A has a C -representation consisting of rank one valuation rings, then A has a unique irredundant finite character representation consisting of rank one valuation rings.

In this section we recover the above results using the topological methods developed in Section 3 and elaborated on in Section 4. Whereas in the articles [2, 3, 44] the strategy is to pass to a maximal Kronecker function ring and treat irredundance there, we work instead with the topology of minimal representations to obtain our results. We also need only that C is a set. A C -representation X is *Noetherian* if X is a Noetherian subspace of \mathfrak{X} with respect to the Zariski topology.

Theorem 5.1. *If $A = (\bigcap_{V \in X} V) \cap C$, where X is a Noetherian subspace of \mathfrak{X} , then $\text{gen}(X)$ contains a Noetherian strongly irredundant C -representation of A .*

Proof. Since X is Noetherian, X is quasicompact, and hence $\text{inv}(X) = \text{gen}(X)$ [45, Proposition 2.2]. Thus by (4.2), there exists a minimal C -representation Y of A in

² The results in [44] also apply to representations consisting of integrally closed rings, not just valuation rings. In light of Finocchiaro's theorem that the space of integrally closed subrings of F is a spectral space (see Example 2.2(6)), it seems likely that this level of generality might be handled with spectral C -representations also.

$\text{gen}(X)$. By Proposition 2.1(1), $\text{gen}(X)$ is a spectral space under the Zariski topology. The elements of $\text{Min}X$ are the maximal elements of $\text{gen}(X)$ under the specialization order of the Zariski topology. In particular, $\text{Min}X \subseteq X$. Thus since $\text{Min}X$ is Noetherian in the Zariski topology, Theorem 2.5 implies that Y is discrete in the inverse topology, so that each valuation ring in Y is an isolated point in Y in the inverse topology. By (4.5), the valuation rings in the minimal C -representation Y are strongly irredundant. Also, by Lemma 2.4, Y is Noetherian in the Zariski topology. \square

Remark 5.2. In general, the strongly irredundant C -representation in Theorem 5.1 is not unique. For example, the uncountably many strongly irredundant representations of the ring A discussed in Example 4.3 are Noetherian spaces in the Zariski topology. The ring A in this case is an overring of a three-dimensional Noetherian domain. By contrast, when C is integrally closed and A is an overring of a two-dimensional Noetherian domain, with A representable by a Noetherian space of valuation overrings, then A has a unique strongly irredundant C -representation [41, Corollary 5.7]. In the case in which A is an overring of a two-dimensional Noetherian domain, the existence of a Noetherian C -representation has strong implications for the structure of A ; see [42].

From the theorem we deduce a corollary that recovers a number of the results discussed at the beginning of the section, with the additional feature that the valuation rings in the representation are strongly irredundant rather than just irredundant. When the valuations are essential, we also obtain uniqueness across all strongly irredundant representations, not just the finite character ones.

Corollary 5.3. *If $A = (\bigcap_{V \in X} V) \cap C$, where X is a finite character subset of \mathfrak{X} , then $\text{gen}(X)$ contains a strongly irredundant C -representation Y of A . If also each valuation ring in X is essential for A , then Y is the only strongly irredundant C -representation of A in \mathfrak{X} .*

Proof. A finite character collection of valuation rings in \mathfrak{X} is Noetherian in the Zariski topology [44, Proposition 3.2], so by Theorem 5.1, $\text{gen}(X)$ contains a strongly irredundant C -representation Y . If also every valuation ring in X is essential for A , then every valuation ring in \mathfrak{X} that does not contain C is C -critical for A (Proposition 4.11), so the assertion of uniqueness follows from (4.13). \square

If A has quotient field F and $A = V \cap R$, where V is a rational valuation overring of A (i.e., V has value group isomorphic to a subgroup of the rational numbers) and R is a subring of F properly containing A , then $V = A_P$, where P is the prime ideal of A that is contracted from the maximal ideal of V [29, Lemma 1.3]. Applying this to the setting of Corollary 5.3, we recover the result of Krull discussed at the beginning of the section, but strengthened to guarantee uniqueness across all strongly irredundant representations.

Corollary 5.4. *Suppose C is a ring and $A = (\bigcap_{V \in X} V) \cap C$, where X is a finite character representation of A consisting of valuation rings of rational rank one. Then X*

contains a strongly irredundant C -representation Y of A , and Y is the only strongly irredundant C -representation of A .

Remark 5.5. Heinzer and Ohm prove a version of Corollary 5.4 for finite character C -representations X of A when A consists of rank one valuation rings (in their terminology, A is a C -domain of finite real character). Thus their approach includes rank one valuation rings with irrational value group also. Unlike rational valuation rings, such valuation rings can be strongly irredundant in X but not essential for A ; see [30, Section 2]. They prove that if A has quotient field F , C is a ring and $A = (\bigcap_{V \in X} V) \cap C$, where X is a finite character subset of \mathfrak{X} consisting of rank one valuation rings, then any valuation ring that is irredundant in some C -representation of A is a member of every finite character C -representation of A that consists of rank one valuation rings, and the collection of all such valuation rings is a C -representation of A [30, Corollary 1.4]. It seems plausible that our approach can recover this result also, but more information is needed about C -representations.

6 Prüfer v -multiplication domains

We assume throughout this section that A is an integrally closed domain a quotient field F , and that \mathfrak{X} is the Zariski-Riemann space of F/A . We no longer work with an intermediate set C , or more precisely, we work with $C = F$. Hence we drop C from our usual terms such as “ C -representation” and “ C -critical” and simply write “representation” for “ F -representation” and “critical” for “ F -critical.”

The concept of a Prüfer v -multiplication domain encompasses that of a Krull domain and a Prüfer domain, as well as polynomial rings over these domains. In this section we apply the point of view developed in Section 4 to the issue of irredundance in representations of Prüfer v -multiplication domains. While the results in this section shed additional light on the domains of Krull type considered in the last section, the real impetus for the section comes from the theory of Prüfer domains. We use the topological methods of Sections 3 and 4 to recover irredundance results for this class of rings as well as generalize them to Prüfer v -multiplication domains.

For an ideal I of A , let $I_v = (A :_F (A :_F I))$. An integral domain A is a v -domain if whenever I, J, K are ideals of A such that $(IK)_v = (JK)_v$, then $I_v = J_v$. Examples of such domains include completely integrally closed domains and Prüfer v -multiplication domains; for a recent survey of this class of rings, see [17]. A v -domain A has a unique maximal Kronecker function ring [22, Theorem 28.1], so by (4.14), A has a unique minimal representation. In particular, A is an intersection of its critical valuation rings.

Theorem 6.1. *A v -domain has at most one strongly irredundant representation.*

Proof. Since a v -domain A is an intersection of critical valuation overrings, we may apply (4.14) in the case where $C = F$ to obtain the theorem. \square

Remark 6.2. In [24, p. 310], Gilmer and Heinzer ask whether it is the case that if A is a v -domain that is an irredundant intersection of valuation rings, then the unique maximal Kronecker function ring of A is an irredundant intersection of valuation rings. We can answer this question in the affirmative under the stipulation that A is represented as a *strongly* irredundant intersection of valuation overrings. In this case, by (4.13), each strongly irredundant representative of A is contained in the minimal representation \mathcal{C} of A consisting of the minimal critical valuation rings for A . By (4.14), $\text{Kr}(\mathcal{C})$ is the unique maximal Kronecker function ring of A . Since \mathcal{C} is a minimal representation of A , (4.7) implies \mathcal{C} contains a dense set of isolated points in the inverse topology. Thus \mathcal{C}^* has a dense set of isolated points, so that by (4.7), $\text{Kr}(\mathcal{C})$ has a strongly irredundant representation.

For each ideal I of A , let $I_t = \sum J$, where J ranges over the finitely generated ideals of A contained in I . An ideal I of A is a *t-ideal* if $I = I_t$. If I is maximal among *t*-ideals, then I is a *t-maximal ideal*. A *t-ideal* I is a *t-prime ideal* if I is prime. A *t-maximal ideal* is a prime, hence *t*-prime, ideal. The set of *t*-prime ideals is denoted $t\text{-Spec } A$, while the set of *t*-maximal ideals is denoted $t\text{-Max } A$. The domain A is a *Prüfer v-multiplication domain (PvMD)* if every nonzero finitely generated ideal of A is *t*-invertible; equivalently, A_M is a valuation domain for each $M \in t\text{-Max } A$ [34, Theorem 3.2]. Every PvMD A is an *essential domain*, meaning that A is an intersection of essential valuation overrings.

In a recent article, Finocchiaro and Tartarone [14, Corollary 2.6] show that an essential domain is a PvMD if and only if the set of its essential valuation overrings is patch closed. We use this characterization to interpret PvMDs in terms of critical valuation rings.

Lemma 6.3. *The domain A is a PvMD if and only if A is an essential domain for which every critical valuation ring of A is essential.*

Proof. Since A is a PvMD, A is an essential domain. By [14, Corollary 2.6], the set E of essential valuation rings is patch closed in the Zariski-Riemann space of A . If $V \in E$, then so is every overring of V , so $\text{gen}(E) = E$. Thus Proposition 2.1(3) implies that E is an inverse closed representation of A , and hence every critical valuation ring of A is in E .

Conversely, suppose every critical valuation ring of A is essential. Then by Proposition 4.11, the set E of essential valuation rings is equal to the set of critical valuation overrings of A . Therefore, E is inverse closed, hence patch closed, so that by [14, Corollary 2.6], A is a PvMD. \square

Theorem 6.4. *Suppose A is a PvMD. Let V be a valuation overring of A , and let P be the center of V in A . Then the following are equivalent.*

- (1) V is strongly irredundant in some representation of A .
- (2) $V = A_P$, $P \in t\text{-Max } A$ and P is isolated in $t\text{-Max } A$ in the inverse topology.
- (3) $V = A_P$, $P \in t\text{-Max } A$ and P contains a finitely generated ideal that is not contained in any other *t*-maximal ideal.

Proof. (1) \Rightarrow (2) Suppose V is strongly irredundant in some representation of A . Since A is a PvMD, A is an essential domain, and hence A is an intersection of its critical valuation rings. Therefore, by (4.13), V is in the collection \mathcal{C} of the valuation rings that are minimal among critical valuation rings for A . By Lemma 6.3, the set of critical valuation rings for A coincides with the set of essential valuation overrings of A . Thus $V = A_P$, and since $V \in \mathcal{C}$, $P \in t\text{-Max } A$. By (4.13), V is strongly irredundant in \mathcal{C} , so that by (4.5), V is isolated in \mathcal{C} with respect to the inverse topology. Since $\mathcal{C} = \{A_Q : Q \in t\text{-Max } A\}$, the map that sends a valuation overring of A to its center in A restricts to a homeomorphism from \mathcal{C} onto $t\text{-Max } A$. (That this map is continuous and closed follows from [51, Lemma 5, p. 119].) Thus P is a t -maximal ideal of A that is isolated in the inverse topology of $t\text{-Max } A$.

(2) \Rightarrow (3) By (2), there is a Zariski quasicompact open subset of $t\text{-Max } A$ whose complement in $t\text{-Max } A$ is $\{P\}$. Statement (3) now follows.

(3) \Rightarrow (2) This is clear.

(2) \Rightarrow (1) As in the proof that (1) implies (2), the canonical map $\mathcal{C} \rightarrow t\text{-Max } A$ is a homeomorphism, so we conclude that V is isolated in the inverse topology on \mathcal{C} . Since \mathcal{C} is a minimal representation of A , (1) follows from (4.5). \square

Remark 6.5. If V is a valuation overring of the domain A that is irredundant in some representation of A as an intersection of valuation overrings and V is essential, then, as discussed after (4.10), V is strongly irredundant. Thus when A is a Prüfer domain, V is irredundant in a representation of A if and only if it is strongly irredundant. Therefore, when A is a Prüfer domain, “strongly irredundant” can be replaced by “irredundant” in Theorem 6.4(1). With this replacement, the theorem recovers a characterization of irredundant representatives of Prüfer domains due to Gilmer and Heinzer; cf. [24, Proposition 1.4 and Lemma 1.6].

In light of Remark 6.5, the next corollary is a version of [24, Theorem 1.10] for PvMDs.

Corollary 6.6. *Suppose A is a PvMD. Then A can be represented (uniquely) as a strongly irredundant intersection of valuation overrings if and only if there is a collection X of t -maximal ideals of A such that*

- (a) *each nonzero element of A is contained in a member of X , and*
- (b) *each $P \in X$ contains a finitely generated ideal that is not contained in any other t -maximal ideal of A .*

Proof. Suppose that A has a strongly irredundant representation Z . By Theorem 6.1 this representation is unique, and by Theorem 6.4 there is a subset X of $t\text{-Max } A$ such that $Z = \{A_P : P \in X\}$. Since $A = \bigcap_{P \in X} A_P$, every nonzero element of A is contained in a member of X . Moreover, by Theorem 6.4, each t -maximal ideal in X contains a finitely generated ideal that is not contained in any other t -maximal ideal. Conversely, if $X \subseteq t\text{-Max } A$ such that (a) and (b) hold for X , then by (a), $Z := \{A_P : P \in X\}$ is a representation of A , and by Theorem 6.4, each $V \in Z$ is strongly irredundant in some representation of A . Therefore, by Theorem 6.1, A has a strongly irredundant representation. \square

Remark 6.7. The ring of integer-valued polynomials $\text{Int}(\mathbb{Z})$ is the set of all polynomials $f(X) \in \mathbb{Q}[X]$ such that $f(\mathbb{Z}) \subseteq \mathbb{Z}$. Among the many well-known properties of this interesting ring is that it is a Prüfer domain; see [5]. In the recent article [6], Chabert and Peruginelli characterize all the rings R between $\text{Int}(\mathbb{Z})$ and $\mathbb{Q}[X]$ that can be represented as an irredundant intersection of valuation overrings. These are precisely the intermediate rings R such that for each prime integer p , the set $\{\alpha \in \widehat{\mathbb{Z}}_p : \mathfrak{M}_{p,\alpha} R \subsetneq R\}$ is dense with respect to the p -adic topology in the ring $\widehat{\mathbb{Z}}_p$ of p -adic integers [6, Remark 5.7]. Here $\mathfrak{M}_{p,\alpha} = \{f \in \text{Int}(\mathbb{Z}) : f(\alpha) \in p\widehat{\mathbb{Z}}_p\}$, a maximal ideal of $\text{Int}(\mathbb{Z})$.

Let A be a PvMD. A *subintersection* of A is an overring of A of the form $\bigcap_{V \in X} V$, where $X \subseteq \{A_P : P \in t\text{-Spec } A\}$. Equivalently, by Lemma 6.3, a subintersection of A is an intersection of critical valuation overrings of A .

Corollary 6.8. *If A is a PvMD such that $t\text{-Spec } A$ is countable, then each subintersection B of A can be represented as a strongly irredundant intersection of essential valuation overrings of A . This representation is the only strongly irredundant representation of B .*

Proof. Let $X = \{A_P : P \in t\text{-Spec } A\}$, and let $Y \subseteq X$. Then $\text{patch}(Y)$ is a representation of $B = \bigcap_{V \in Y} V$. Since by [14, Corollary 2.6], X is a patch closed representation of A , $\text{patch}(Y) \subseteq X$, and hence $\text{patch}(Y)$ is countable. Therefore, by (4.8), B has a strongly irredundant representation in $\text{patch}(Y)$. Since the valuation rings in X are essential for A , hence essential for B , Proposition 4.11 implies that the valuation rings in Y are critical for B . Thus the ring B has a strongly irredundant representation consisting of critical valuation rings. As a subintersection of A , B is a Prüfer v -multiplication domain, hence a v -domain. Therefore, by Theorem 6.1, there is only one strongly irredundant representation of B . \square

When A is a PvMD for which $t\text{-Max } A$ is a Noetherian subspace of $\text{Spec } A$, then $\{A_P : P \in t\text{-Max } A\}$ is a Noetherian space of valuation overrings that represents A . Therefore, by Theorem 5.1, A has a strongly irredundant representation in $\{A_P : P \in t\text{-Max } A\}$. However, the fact that A is a PvMD allows us to assert the stronger claim that $\{A_P : P \in t\text{-Max } A\}$ itself is a strongly irredundant representation of A , and that this property is inherited by similarly constituted representations of subintersections of A . We prove this in the next theorem.

Theorem 6.9. *Suppose A is a PvMD. Then $t\text{-Max } A$ (resp., $t\text{-Spec } A$) is Noetherian in the Zariski topology if and only if each collection X of incomparable essential valuation overrings is a strongly (resp., tightly) irredundant representation of $\bigcap_{V \in X} V$.*

Proof. Suppose $t\text{-Max } A$ is a Noetherian space. By [14, Corollary 2.6], the set $E = \{A_P : P \in t\text{-Spec } A\}$ of essential valuation overrings of A is patch closed. Thus by Proposition 2.1(4)(a), E is spectral in the Zariski topology. Since $\text{Min } E = \{A_P : P \in t\text{-Max } A\}$ is by assumption Noetherian in the Zariski topology, Theorem 2.5 implies every subset X of E consisting of incomparable valuation rings is discrete in the inverse topology.

Now let X be a subset of E consisting of incomparable valuation rings. The ring $B := \bigcap_{V \in X} V$ is again a PvMD [34, Corollary 3.9]. We claim that $X = \{B_P : P \in t\text{-Max } B\}$. To this end, let P be a maximal t -ideal of B . Then since by Lemma 2.4, X is Noetherian and $B = \bigcap_{V \in X} V$, we have $B_P = \bigcap_{V \in X} (VB_P)$ [44, Theorem 3.5] and $\{VB_P : V \in X\}$ is a Noetherian subspace of \mathfrak{X} in the Zariski topology [44, Theorem 3.7]. Thus $\{VB_P : V \in X\}$ satisfies DCC, and since $B_P = \bigcap_{V \in X} (VB_P)$ is a valuation ring, this forces $B_P = VB_P$ for some $V \in X$, and hence $V \subseteq B_P$. Since V is essential for A , hence for B , $V = B_Q$ for some t -prime ideal Q of B . Therefore, since P is a t -maximal ideal and $B_Q \subseteq B_P$, we have $B_P = V \in X$, which shows $\{B_P : P \in t\text{-Max } B\} \subseteq X$.

To verify the reverse inclusion, let $W \in X$. Then since W is essential, $W = B_Q$ for some prime ideal Q in B , and hence Q is a t -prime ideal of B . Let L be a maximal t -ideal of B containing Q . Then, by what we have shown, $B_L \in X$, so since $B_L \subseteq W$ and the members of X are incomparable, it must be that $B_L = W = B_Q$, proving that $Q \in t\text{-Max } B$. This proves that $X = \{B_P : P \in t\text{-Max } B\}$. Therefore, it follows from [51, Lemma 5, p. 119] that X is homeomorphic to $t\text{-Max } B$, so that $t\text{-Max } B$ is discrete in the inverse topology. By Theorem 6.4 each member of X is strongly irredundant in some representation of B . By Proposition 4.11, the valuation rings in X are critical for B , so by (4.14), X is a strongly irredundant representation of B .

Conversely, suppose each collection X of incomparable valuation rings in $\{A_P : P \in t\text{-Spec } A\}$ is a strongly irredundant representation of $\bigcap_{V \in X} V$. Then by Theorem 6.4 each such collection X is discrete in the inverse topology, and consequently, each collection of incomparable prime ideals in $t\text{-Spec } A$ is discrete in the inverse topology. Therefore, by Theorem 2.5, $t\text{-Max } A$ is Noetherian in the Zariski topology.

Now suppose that $t\text{-Spec } A$ is Noetherian. As noted above, $E = \{A_P : P \in t\text{-Spec } A\}$ is a spectral space; it is also Noetherian since $t\text{-Spec } A$ is. Therefore, for any $V \in E$, the set $\{W \in E : V \subseteq W\}$ satisfies DCC. Thus if X is any collection of incomparable valuation rings in E , since we have established that each valuation ring in X is strongly irredundant in the intersection $\bigcap_{V \in X} V$, it follows that each valuation ring in X is tightly irredundant.

Conversely, suppose that each collection X of incomparable critical valuation overrings is a tightly irredundant representation of $\bigcap_{V \in X} V$. We have established already that this implies that $t\text{-Max } A$ is Noetherian. Thus by Lemma 2.4, to prove that $t\text{-Spec } A$ is Noetherian, we need only verify that for each $V \in E$, the set $\{W \in E : V \subsetneq W\}$ has a minimal element with respect to \subseteq . Let $V \in E$. Then by assumption V is tightly irredundant in the representation $\{V\}$ of the ring V , so the intersection of the valuation rings in $\{W \in E : V \subsetneq W\}$ is again in this same set. Therefore, $t\text{-Spec } A$ is Noetherian. \square

Gabelli, Houston and Lucas [20] define a domain A to have property $(t\#)$ if whenever X is a collection of pairwise incomparable t -prime ideals and $P \in X$, $\bigcap_{Q \neq P} A_Q \subsetneq A_P$, where Q ranges over the prime ideals in X distinct from P . Using Theorem 6.9, additional characterizations of PvMDs with Noetherian t -maximal spectrum can be deduced from the work of Gabelli, Houston and Lucas; see for example Propositions 2.4, 2.6 and 2.8 of [20]. Similarly, for a PvMD A , the characterization of Noetherian spectral spaces given in Lemma 2.3 can be used to link

the property in which $t\text{-Spec } A$ is Noetherian to the work of Gabelli, Houston and Lucas, specifically to the equivalent characterizations in Propositions 2.11 and Theorem 2.14 in [20]. For additional applications, see [4]. For example, it follows from Corollary 2.6 and [4, Theorem 3.9] that A is a generalized Krull domain if and only if A is a PvMD for which $t\text{-Max } A$ is Noetherian and $P \neq (P^2)_t$ for each t -prime ideal P of A .

The $(t\#)$ property extends to non-Prüfer settings the property $(\#)$ introduced for Prüfer domains by Gilmer [21] and studied further by Gilmer and Heinzer in [23]. A Prüfer domain A is said to satisfy $(\#)$ if for each maximal ideal M of A , A_M is irredundant in the representation $\{A_N : N \in \text{Max } A\}$ of A . Property $(\#)$ and the stronger version $(\#\#)$, which requires that every overring has $(\#)$, play an important role in the local-global theory of Prüfer domains; see for example [15, 16]. Since every maximal ideal of a Prüfer domain is a t -maximal ideal, the properties $(\#)$ and $(t\#)$ coincide for Prüfer domains. Thus we have the following topological characterization of Prüfer domains satisfying $(\#\#)$. The corollary, which is immediate from Theorem 6.9, is implicit in [19, Theorem 5.14], where it is proved using the work of Rush and Wallace [48].

Corollary 6.10. *Suppose A is Prüfer domain. Then $\text{Max } A$ is a Noetherian space if and only if A satisfies $(\#\#)$. \square*

7 Irredundance in intersections of irreducible ideals

An ideal A of the ring R is *irreducible* if A is not an intersection of two ideals properly containing it. Every ideal A is the intersection of the irreducible ideals containing it. Indeed, if $r \in R \setminus A$, then by Zorn's Lemma, there exists an ideal of R maximal with respect to containing A and not containing r . This ideal is necessarily irreducible, from which it follows that A is an intersection of irreducible ideals. In this section we are interested in when A is an irredundant intersection of irreducible ideals. We show how the topological approach of Section 3 can be applied to intersection decompositions of irreducible ideals in *arithmetical rings*, those rings R for which the ideals of R_M form a chain for each maximal M of R . The ambient spectral space from which these intersection representations are drawn is $\text{Irr } R$, the set of proper irreducible ideals of R , viewed as a topological space having a basis of closed sets of the form $\mathcal{V}(A) := \{B \in \text{Irr } R : A \subseteq B\}$, where A ranges over all the ideals of R . Since $R \notin \text{Irr } R$, the maximal elements in $\text{Irr } R$ are the maximal ideals of R . By a *representation* of A we mean a subset of $\text{Irr } R$ whose intersection is A .

Lemma 7.1. *Let R be an arithmetical ring. Then for each proper ideal A of R , $\mathcal{V}(A)$ is a spectral representation of A that does not contain any proper closed representations of A .*

Proof. Every irreducible ideal in an arithmetical ring is strongly irreducible [31, Lemma 2.2(3)]. Also, the intersection of two finitely generated ideals in an arithmetical ring is finitely generated [50, Corollary 1.11], so by Example 2.2(4), $\mathcal{V}(A)$

is a spectral representation of A . Finally, suppose X is a closed subset of $\mathcal{V}(A)$ that is a representation of A . Then $X = \mathcal{V}(B)$ for some proper ideal B of R . Since every proper ideal of R is an intersection of irreducible ideals, B is the intersection of the ideals in $\mathcal{V}(B)$. Since $\mathcal{V}(B)$ is a representation of A , this forces $A = B$. Therefore, no proper closed subset of $\mathcal{V}(A)$ is a representation of A . \square

Let A be a proper ideal of the arithmetical ring R . Since $\mathcal{V}(A)$ is a spectral space and the specialization order agrees with set inclusion, $\mathcal{V}(A)$ contains minimal elements with respect to set inclusion and A is an intersection of these minimal irreducible ideals. Using the ideas developed in Section 3, along with the results about Noetherian spectral spaces in Section 2, we obtain a version of a theorem proved in [19] by different methods. We recall that a few notions from [19]. A *Krull associated prime* of an ideal A of a ring R is a prime ideal that is a union of ideals of the form $A : r = \{s \in R : rs \in A\}$ with $r \in R$. If P is a prime ideal of R , we set $A_{(P)} := \{r \in R : br \in A \text{ for some } b \in R \setminus P\}$. A Zorn's Lemma argument shows that the set of Krull associated primes of A contains maximal elements. Let \mathcal{X}_A denote the set of these maximal elements. Then (with R arithmetical) we have $\{A_{(P)} : P \in \mathcal{X}_A\} = \text{Min } \mathcal{V}(A)$; see [19, Theorem 5.8].

Theorem 7.2. (cf. [19, Theorem 5.14]) *If R is an arithmetical ring for which $\text{Max } R$ is Noetherian in the Zariski topology, then for each proper ideal A of R the set of irreducible ideals that are minimal over A is a strongly irredundant representation of A , and this is the unique irredundant representation of A as an intersection of irreducible ideals.*

Proof. Let A be a proper ideal of R . As discussed before the theorem, $\text{Min } \mathcal{V}(A) = \{A_{(P)} : P \in \mathcal{X}_A\}$. Let $P \in \mathcal{X}_A$. Then by Lemma 2.3 there is a finitely generated ideal $I \subseteq P$ such that I is not contained in any maximal ideal that does not contain P . Since R is an arithmetical ring, the prime ideals of R form a tree under inclusion, so the only prime ideal in \mathcal{X}_A that contains I is P . Thus since P is a union of ideals of the form $A : r$, $r \in R$, there are $r_1, \dots, r_n \in R$ such that P is the only ideal in \mathcal{X}_A containing $(A : r_1) + \dots + (A : r_n)$. Since R is arithmetical, the latter ideal is equal to $A : (r_1 R \cap \dots \cap r_n R)$ [37, Exercise 18(c), p. 151]. Hence $r_1 R \cap \dots \cap r_n R \subseteq (\bigcap_{Q \neq P} A_{(Q)}) \setminus A_{(P)}$, where Q ranges over the prime ideals in $\mathcal{X}_A \setminus \{P\}$. This shows that the representation $\text{Min } \mathcal{V}(A) = \{A_{(P)} : P \in \mathcal{X}_A\}$ of A is irredundant.

Next, since by Lemma 7.1, $\mathcal{V}(A)$ is a minimal closed representation of A , we have by Theorem 3.6 that $\text{Min } \mathcal{V}(A)$ is a strongly irredundant representation of A . Finally, since $\mathcal{V}(A)$ is a minimal closed representation of A , all the irreducible ideals in $\mathcal{V}(A)$ are critical for A in the spectral representation $\mathcal{V}(A)$ of A . Therefore, by Corollary 3.12, every irredundant representation of A is strongly irredundant, and hence by Theorem 3.13(4), there is a unique irredundant representation of A . \square

Remark 7.3. By [19, Theorem 5.14], the converse of Theorem 7.2 is also true: If every ideal A of a ring R can be written uniquely as an irredundant intersection of the irreducible ideals that are minimal with respect to containing A , then R is an arithmetical ring with Noetherian maximal spectrum.

Remark 7.4. The ideas in Section 3 can also be applied to the intersections of prime ideals. Let A be a radical ideal of a ring R . Then $\mathcal{V}(A) = \{P \in \operatorname{Spec} R : A \subseteq P\}$ is a spectral representation of A , each member of which is critical for A . Thus A has a strongly irredundant representation if and only if the set of minimal primes of R/A contains a dense set of isolated points with respect to the Zariski topology (Corollary 3.7 and Theorem 3.13). Also, every irredundant representative of A is strongly irredundant (Corollary 3.12), and there is at most one irredundant representation of A (Theorem 3.13). Finally, if $\mathcal{V}(A)$ is countable, then A has a strongly irredundant representation (Corollary 3.9).

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